Eur. Phys. J. C 19, 517–528 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100629

THE EUROPEAN PHYSICAL JOURNAL C

© Società Italiana di Fisica
Springer-Verlag 2001

Comparison of πK scattering in $SU(3)$ chiral perturbation theory and dispersion relations*

B. Ananthanarayan[1], P. Büttiker[2]

[1] Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India
[2] Institut für Kernphysik, Forschungszentrum Jülich, 52425 Jülich, Germany

Received: 6 December 2000 / Revised version: 13 February 2001 /
Published online: 23 March 2001 – © Springer-Verlag 2001

Abstract. We establish the framework for the comparison of πK scattering amplitudes from $SU(3)$ chiral perturbation theory with suitable dispersive representations which result from the combination of certain fixed-t dispersion relations with dispersion relations on hyperbolic curves. This allows for predictions for some combinations of low energy constants appearing in higher order calculations of chiral perturbation theory. Using a simple parametrization for the lowest partial waves, first estimates for some combinations are presented.

1 Introduction

The pseudoscalar octet of pions, kaons and the η may be viewed as the Goldstone bosons of the spontaneously broken approximate symmetry of the QCD Lagrangian whose interactions may be described by $SU(3)$ chiral perturbation theory [1]. The πK scattering amplitudes have been computed in this framework sometime ago, see, [2,3]. For an extensive review of phenomenological information prior to these developments, including dispersion relation analysis, we refer to [4]. Our aim here is to set up the appropriate framework within which the chiral amplitudes can be compared with dispersive representations of the amplitudes, of the type established in axiomatic field theory.

It is instructive to first discuss $\pi\pi$ scattering which has been studied in considerable detail. The $\pi\pi$ scattering amplitude has been computed to one-loop accuracy [5], and to two loops in $SU(2)$ chiral perturbation theory [6,7], and at one-loop accuracy in $SU(3)$ chiral perturbation theory [2]. In $SU(2)$ chiral perturbation theory up to two-loop accuracy, the amplitude is described by three functions of a single (Mandelstam) variable, whose absorptive parts are given in terms of those of the three lowest partial waves [8]. One replaces them by an dispersive representation which yields an amplitude with an effective low energy polynomial and a dispersive tail [6,9]. A dispersion relation representation with two subtractions is an ideal starting point for rewriting them in a form whereby a comparison with the chiral representation can be made, when the S– and P–wave absorptive parts alone are retained. The absorptive parts of the higher waves contribute to the polynomial pieces only. We note that the analysis in the past has been performed only for elastic scattering in $SU(2)$ chiral perturbation theory; we extend it in a straightforward manner to $SU(3)$ chiral perturbation theory where $K\overline{K}$ and $\eta\eta$ are present in intermediate states. The remarkable synthesis of dispersion relation phenomenology (see [10] and references therein) and chiral perturbation theory has recently led to a highly accurate prediction for the iso-scalar S–wave scattering length [11].

Recently, there has been an revival in the interest in πK scattering. There are indications for a flavour dependence of the size of the quark condensate [12]. As πK scattering is the most simple $SU(3)$–process involving kaons, it is the suitable place to test this dependence. Furthermore, there are plans to measure the lifetime of πK–atoms to an accuracy of 20% in the DIRAC experiment at CERN [13]. This would allow for a very precise determination of the scattering length a_0^- within an error of 10%. The role of the latter must not be understated: one may show that in $SU(3)$ chiral perturbation theory to one loop the scattering length a_0^- depends only on the constant L_5^r and the pion and the kaon decay constant. Furthermore, L_5^r is itself determined, at this order, by the ratio of these two decay constants, see [1], leaving a_0^- free of any low energy constants. This observation is being published here for the first time, although it was already known [14] to the authors of [2]. Note that a similar statement holds for the tree–level prediction of a_0^- in generalized chiral perturbation theory, where the *a priori* unknown low energy constants appear even at tree–level [15]. Therefore, a comparison of experimental values of a_0^- with its theoretical predictions from chiral perturbation theory is a stringent test of the framework of chiral symmetry breaking. Apart

* Work supported in part by DFG under contract no. ME 864-15/2

from that, an independent high precision estimate for this scattering length calls for a fresh partial wave analysis since this threshold parameter can play the role of a subtraction constant in πK dispersion relations.

The structure of the πK amplitudes is best revealed when we consider a system of amplitudes defined by $T^+(s,t,u)$ and $T^-(s,t,u)$, which are even and odd under the interchange of s and u, respectively. We demonstrate that each of these chiral amplitudes may be written down in terms of three functions of single variables whose absorptive parts are related to those of the S– and P–waves in the s– and t–channels.

The dispersion relations we employ for T^+ are the ones as given in [16] where fixed-t dispersion relations are combined with dispersion relations on hyperbolic curves. We introduce a new dispersion relation here for T^-. Retaining only the absorptive parts of the S– and P–waves we demonstrate the equivalence of the structure of the representation to that of the chiral result. This allows us to match the representations in the low energy domain, after adding the contributions of the higher waves which are only polynomials to this order. Since πK scattering at low energies is dominated by the S– and P–waves, a detailed knowledge of these waves is important. Stringent constraints resulting from axiomatic analyticity and crossing are best expressed in terms of integral (Steiner-Roy) equations [4,17–19]. These equations are the ideal starting point for a future analysis of πK scattering information. Due to the importance of these equation in such an analysis, the Steiner-Roy equations in the S- and P-wave approximation for the S- and P-waves are given here explicitly. (It may be noted that one can proceed to analyze πN scattering [20] in an analogous manner.)

While an accurate phase shift parametrization, independent from input from chiral perturbation theory is awaited, we use a simple K-matrix approach to saturate the dispersion relations which then provide the first estimates for certain combinations of the low energy constants of $SU(3)$ chiral perturbation theory from πK scattering.

The plan of the paper is as follows: in Sect. 2 we establish our notation and conventions, in Sect. 3 we carry out the decomposition of the one-loop πK amplitudes into functions of single variables and then discuss the method of replacing them with a dispersive representation. In Sect. 4 the dispersion relations are considered and rewritten in the S– and P–wave absorptive part approximation, the contributions of the higher waves are discussed, and the comparison with the chiral amplitude is outlined. Furthermore, we explicitly give the Roy equations for the lowest πK partial waves. In Sect. 5 we saturate the dispersion relations with phenomenological absorptive parts and discuss the consequences. In Sect. 6 we provide a discussion and a summary of the results. Appendix A briefly summarizes the results obtained when applying the matching mentioned above to $SU(3)$ $\pi\pi$ scattering, in Appendix B functions of single variables of interest are listed, and in Appendix C the kernels of the Steiner-Roy integral equations for the S– and P–waves are given.

2 Notation and conventions

We consider the process

$$\pi^{I_1}(p_1) + K^{J_1}(q_1) \to \pi^{I_2}(p_2) + K^{J_2}(q_2),$$

with the four-momenta p_i, q_i and the isospin I_i and J_i of the pions and the kaons, respectively. The Mandelstam variables are defined as ($\Sigma \equiv M^2 + m^2$)

$$s = (p_1+q_1)^2,\ t = (q_1-q_2)^2,\ u = (q_1-p_2)^2,$$

with

$$s + t + u = 2\Sigma,$$

where M and m are the pion and the kaon mass, respectively. In the s-channel the center of mass scattering angle Θ_s and momentum q_s are given by ($\Delta \equiv M^2 - m^2$)

$$z_s \equiv \cos\Theta_s = 1 + \frac{t}{2q_s^2} = \frac{t - u + \frac{\Delta^2}{s}}{4q_s^2},$$
$$q_s^2 = \frac{(s-(m-M)^2)(s-(m+M)^2)}{4s},$$

and the partial wave decomposition is defined by

$$T^{I_s}(s,t,u) = 16\pi \sum (2l+1) f_l^{I_s}(s) P_l(z_s).$$

The partial waves may then be parametrized by the phase shifts δ_l^I and the inelasticities η_l^I,

$$f_l^I(s) = \frac{\sqrt{s}}{2q_s}\frac{1}{2i}\left\{\eta_l^I(s)e^{2i\delta_l^I(s)} - 1\right\},$$

and have the threshold expansion

$$\mathrm{Re}\, f_l^I(s) = \frac{\sqrt{s}}{2} q^{2l}\left\{a_l^I + b_l^I q^2 + O(q^4)\right\}.$$

In the t-channel, the center of mass momenta of the pion and the kaon are q_t and p_t, respectively, and the centre of mass scattering angle Θ_t is given by

$$z_t \equiv \cos\Theta_t = \frac{s + p_t^2 + q_t^2}{2q_t p_t} = \frac{s-u}{4p_t q_t},$$
$$p_t = \sqrt{\frac{t-4m^2}{4}},\ q_t = \sqrt{\frac{t-4M^2}{4}}.$$

The partial waves are defined by

$$T^{I_t}(s,t,u) = 16\pi\sqrt{2}\sum(2l+1) f_l^{I_t}(t) P_l(z_t).$$

Once one of the isospin amplitudes is known the other and combinations of these are fixed by crossing symmetry:

$$T^{1/2}(s,t,u) = \frac{3}{2}T^{3/2}(u,t,s) - \frac{1}{2}T^{3/2}(s,t,u),$$
$$T^+(s,t,u) \equiv \frac{1}{3}T^{1/2}(s,t,u) + \frac{2}{3}T^{3/2}(s,t,u),$$

$$= \frac{1}{\sqrt{6}} T^{I_t=0}(s,t,u),$$
$$T^-(s,t,u) \equiv \frac{1}{3}T^{1/2}(s,t,u) - \frac{1}{3}T^{3/2}(s,t,u)$$
$$= \frac{1}{2}T^{I_t=1}(s,t,u)$$

It may be seen from the above that $T^+(s,t,u)$ is even under the interchange of s and u, whereas $T^-(s,t,u)$ is odd.

3 Decomposition of the chiral amplitudes

In the framework of one-loop $SU(3)$ chiral perturbation theory, the explicit expression for $T^{3/2}(s,t,u)$ has been presented in [2]. One then constructs the two amplitudes of interest $T^+(s,t,u)$ and $T^-(s,t,u)$. It may be seen that these can now be decomposed into terms involving functions of single variables only:

$$\begin{aligned} T^+(s,t,u) =& Z_0^+(s) + Z_0^+(u) + \left(t - s + \frac{\Delta^2}{u}\right) Z_1^+(u) \\ &+ \left(t - u + \frac{\Delta^2}{s}\right) Z_1^+(s) + Z_t^+(t), \\ T^-(s,t,u) =& Z_0^-(s) - Z_0^-(u) + \left(t - s + \frac{\Delta^2}{u}\right) Z_1^-(u) \\ &- \left(t - u + \frac{\Delta^2}{s}\right) Z_1^-(s) + (s-u) Z_t^-(t). \end{aligned} \quad (1)$$

Written in this form, the imaginary parts of the Z's are related to those of the lowest partial waves in the following manner $(s \geq (m+M)^2, t \geq 4M^2)$:

$$\begin{aligned} \mathrm{Im}\, Z_0^\pm(s) &= 16\pi\, \mathrm{Im}\, f_0^\pm(s), \\ \mathrm{Im}\, Z_1^\pm(s) &= \frac{12\pi}{q_s^2} \mathrm{Im}\, f_1^\pm(s), \\ \mathrm{Im}\, Z_t^+(t) &= \frac{16\pi}{\sqrt{3}} \mathrm{Im}\, f_0^{I_t=0}(t), \\ \mathrm{Im}\, Z_t^-(t) &= 6\sqrt{2}\pi\, \mathrm{Im}\, \frac{f_1^{I_t=1}(t)}{p_t q_t}. \end{aligned} \quad (2)$$

In Appendix B we present our choice of $Z_i^\pm$, $i = 0, 1, t$[1]. The imaginary parts of $Z_0^\pm$ and $Z_1^\pm$ receive contributions from the πK and $K\eta$ loops with the lower cut starting at the πK threshold $s = (M+m)^2$. On the other hand the imaginary parts of $Z_t^\pm$ receive contributions from the $\pi\pi$ and $K\bar{K}$ loops and Z_t^+ alone from $\eta\eta$ loops, with the lowest cut starting at the $\pi\pi$ threshold $s = 4M^2$. The former when written out in terms of amplitudes of definite iso-spin in the $s-$ channel are such that they respect the elastic unitarity condition

$$\mathrm{Im} f_l^I(s) = \frac{2q_s}{\sqrt{s}} |f_l^I(s)|^2.$$

[1] The decomposition does not uniquely fix the algebraic parts of the functions, which is a consequence of not all the Mandelstam variables being independent.

The latter respect the principle of extended unitarity, *viz.*

$$\arg f_0^{I_t=0}(t) = \delta_0^0(t)(\pi\pi), \quad \arg f_1^{I_t=1}(t) = \delta_1^1(t)(\pi\pi),$$
$$\text{if} \quad 4M^2 \leq t \leq 4m^2.$$

Keeping this in mind, and using (2), it can be shown that the $Z_i^\pm$, $i = 0, 1, t$ verify the following relations (written out to enable a comparison of the chiral and dispersive amplitudes to this order in chiral perturbation theory):

$$\begin{aligned} Z_0^\pm(s) &= \frac{\alpha_0^\pm}{s} + \beta_0^\pm + \gamma_0^\pm s + \delta_0^\pm s^2 \\ &\quad + 16 s^3 \int_{(m+M)^2}^\infty \frac{ds'}{s'^3} \frac{\mathrm{Im}\, f_0^\pm(s')}{s'-s}, \\ Z_1^\pm(s) &= \beta_1^\pm + \gamma_1^\pm s + 12 s^2 \int_{(m+M)^2}^\infty \frac{ds'}{s'^2} \frac{1}{q_{s'}^2} \frac{\mathrm{Im}\, f_1^\pm(s')}{s'-s}, \\ Z_t^+(t) &= \beta_t^+ + \gamma_t^+ t + \delta_t^+ t^2 + \frac{16 t^3}{\sqrt{3}} \int_{4M^2}^\infty \frac{dt'}{t'^3} \frac{\mathrm{Im}\, f_0^{I_t=0}(t')}{t'-t}, \\ Z_t^-(t) &= \beta_t^- + \gamma_t^- t + 6\sqrt{2} t^2 \int_{4M^2}^\infty \frac{dt'}{t'^2} \frac{1}{t'-t} \mathrm{Im}\, \frac{f_1^{I_t=1}(t')}{p_{t'} q_{t'}}. \end{aligned} \quad (3)$$

The subtraction constants $\alpha_i^\pm, \beta_i^\pm, \gamma_i^\pm$, and $\delta_i^\pm$ depend on the low energy constants L_i^r and may be simply evaluated from the explicit expressions we have provided for the $Z_i^\pm$, $i = 0, 1, t$. Note that the appearance of the poles in $Z_0^\pm$ is due to the unequal masses of the particles. However, they cancel the kinematic poles appearing in the coefficients of $Z_1^\pm$ such that in the chiral representation (1) these poles disappear. With (3) and (1) we may write

$$\begin{aligned} &T^+(s,t,u) = \\ &2\,\beta_0^+ + \beta_t^+ - 2\,\left(m^4 + 6\,m^2\,M^2 + M^4\right)\,\gamma_1^+ \\ &\quad + (s+u)\,\left(\gamma_0^+ - \beta_1^+\right) + \left(s^2 + u^2\right)\,\left(\gamma_1^+ + \delta_0^+\right) \\ &\quad + t\,\left(2\,\beta_1^+ + 6\,\left(m^2 + M^2\right)\,\gamma_1^+ + \gamma_t^+\right) + t^2\,\left(\delta_t^+ - 2\,\gamma_1^+\right) \\ &\quad + 16 \int_{(m+M)^2}^\infty \frac{d\,s'}{s'^3} \left[\frac{s^3}{s'-s} + \frac{u^3}{s'-u}\right] \mathrm{Im} f_0^+(s') \\ &\quad + \frac{16}{\sqrt{3}} t^3 \int_{4M^2}^\infty \frac{d\,t'}{t'^3} \frac{\mathrm{Im}\, f_0^{I_t=0}(t')}{t'-t} \\ &\quad + 12\,s^2 \left(t - u + \frac{\Delta^2}{s}\right) \int_{(m+M)^2}^\infty \frac{d\,s'}{s'^2} \frac{1}{q_{s'}^2} \frac{\mathrm{Im}\, f_1^+(s')}{s'-s} \\ &\quad + 12\,u^2 \left(t - s + \frac{\Delta^2}{u}\right) \int_{(m+M)^2}^\infty \frac{d\,s'}{s'^2} \frac{1}{q_{s'}^2} \frac{\mathrm{Im}\, f_1^+(s')}{s'-u}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} &T^-(s,t,u) = \\ &\left(\beta_1^- + \beta_t^- + \gamma_0^-\right)\,(s-u) + \left(\gamma_1^- + \gamma_t^-\right)\,t\,(s-u) \\ &\quad + \delta_0^-\,\left(s^2 - u^2\right) + 16 \int_{(m+M)^2}^\infty \frac{d\,s'}{s'^3} \left[\frac{s^3}{s'-s} - \frac{u^3}{s'-u}\right] \\ &\quad \times \mathrm{Im}\, f_0^-(s') + 6\,\sqrt{2}\,t^2 (s-u) \int_{4M^2}^\infty \frac{d\,t'}{t'^2 (t'-t)} \end{aligned}$$

$$\times \mathrm{Im}\, \frac{f_1^{I_t=1}(t')}{q_{t'}p_{t'}} + 12s^2 \left(t-u+\frac{\Delta^2}{s}\right) \int_{(m+M)^2}^{\infty} \frac{d\,s'}{s'^2} \times \frac{1}{q_{s'}^2} \frac{\mathrm{Im}\, f_1^-(s')}{s'-s} - 12u^2 \left(t-s+\frac{\Delta^2}{u}\right) \int_{(m+M)^2}^{\infty} \frac{d\,s'}{s'^2} \times \frac{1}{q_{s'}^2} \frac{\mathrm{Im}\, f_1^-(s')}{s'-u}. \quad (5)$$

The polynomial part of $T^{\pm}$ then reads[2], after eliminating the ambiguity associated with the mass shell condition $s+t+u=2\Sigma$ (see footnote 1):

$$\begin{aligned} &T^+_{P,\chi}(s,t,u) = \\ &\Big\{2\beta_0^+ + \beta_t^+ + 2\Big[\Delta^2\gamma_1^+ - \Sigma(\beta_1^+ - \gamma_0^+) \\ &\quad +\Sigma^2(\delta_0^+ - \gamma_1^+)\Big]\Big\} + \Big\{3\beta_1^+ - \gamma_0^+ + \gamma_t^+ \\ &\quad -2\Sigma(\delta_0^+ - 2\gamma_1^+)\Big\}t + \frac{1}{2}(\delta_0^+ - 3\gamma_1^+ + 2\delta_t^+)t^2 \\ &\quad +\frac{1}{2}(\delta_0^+ + \gamma_1^+)(s-u)^2, \\ &T^-_{P,\chi}(s,t,u) = \\ &(\beta_1^- + \beta_t^- + \gamma_0^- + 2\Sigma\delta_0^-)(s-u) \\ &\quad +(\gamma_1^- + \gamma_t^- - \delta_0^-)(s-u)t. \end{aligned} \quad (6)$$

It might be noted that chiral perturbation theory could provide an accurate description of the πK scattering amplitude in the low-energy domain, if we could compare the representation given above with a suitable representation provided by dispersion relations, upon exploiting analyticity and crossing properties of the amplitudes. Furthermore, it is the six lowest partial waves that essentially determine the low-energy structure completely and also fix the low energy constants when the chiral and dispersive representations are compared, up to some unknown subtraction constants, a role that is played by the scattering lengths. These partial waves, in principle, are related through analyticity and crossing by integral equations that are generated by the dispersion relations for the amplitudes T^+ and T^-. In the next section it is precisely those dispersion relations which provide this framework which are first set up and analyzed and then used to generate the system of integral equations. When the absorptive parts of all $l \geq 2$ waves are neglected, the system of equations is a closed system of equations for these waves and imposing unitarity on the partial waves constrains them further. Such a system could be used in the future for an analysis of presently available and forthcoming data to pin down the scattering lengths within relatively small uncertainties and to determine the low energy constants through a program of sum rules.

Note that the πK scattering amplitude at tree-level in generalized chiral perturbation theory has been discussed in [15] and in the heavy-kaon effective theory [21]. Our methods can be extended to analyze these theories as well.

[2] The explicit expressions for $\alpha_i^{\pm}, \beta_i^{\pm}, \gamma_i^{\pm}$, and $\delta_i^{\pm}$ may be obtained from the authors.

4 Dispersion relations for πK scattering

In field theory the scattering amplitudes T^+ and T^- verify fixed-t dispersion relations, under conventional assumptions regarding the high energy behaviour, the former with two subtractions and latter with none. In practice, we have found that in order to meet the requirements of matching the chiral expansion with the axiomatic representation, dispersion relations with two subtractions for T^- as well prove to be convenient. In [16], the unknown t- dependent subtraction function was eliminated by considering dispersion relations on a certain hyperbola, $s \cdot u = \Delta^2$, resulting in a representation that we find most suitable for our purposes. The primary reason for this is that it is the choice of comparing the fixed-t dispersion relations and the hyperbolic dispersion relations on the hyperbola given above and at $t=0$ which ensures that the role of the subtraction constant is played by the scattering length (see below). A different choice would have led to the value of the scattering amplitude at a kinematic point that does not correspond to the threshold to be the effective subtraction point. The fixed-t dispersion relation for T^+ is given by

$$\begin{aligned} &T^+(s,t,u) = \\ &8\pi(m+M)a_0^+ + \frac{1}{\pi}\int_{(m+M)^2}^{\infty} \frac{d\,s'}{s'^2}\left[\frac{s^2}{s'-s} + \frac{u^2}{s'-u}\right] \\ &\quad \times A_s^+(s',t) + S^+ + L^+(t) + U^+(t). \end{aligned} \quad (7)$$

The expressions for $S^+, L^+(t)$, and $U^+(t)$ can be found in [16], which when adopted to our normalization conventions for the amplitude are

$$\begin{aligned} S^+ &= \frac{1}{2\pi}\int_{(m+M)^2}^{\infty} d\,s' \frac{\Delta^2 - s'\Sigma}{q_{s'}^2 s'^2} A_s^+(s', t'_{\Delta^2}), \\ L^+(t) &= \frac{t}{\pi}\int_{4M^2}^{\infty} \frac{d\,t'}{t'(t'-t)} A_t^+(t', \Delta^2), \\ U^+(t) &= \frac{1}{\pi}\int_{(m+M)^2}^{\infty} d\,s' \frac{s'(2\Sigma - t) - 2\Delta^2}{s'^2(4q_{s'}^2 + t)} A_s^+(s', t'_{\Delta^2}) \\ &\quad -\frac{1}{\pi}\int_{(m+M)^2}^{\infty} d\,s' \\ &\quad \times \frac{s'(2\Sigma - t)^2 - 2\Delta^2 s' - \Delta^2(2\Sigma - t)}{s'^3(4q_{s'}^2 + t)} A_s^+(s', t). \end{aligned}$$

It is important to keep in mind that the absorptive parts $A_s^+(s', t'_{\Delta^2})$ and $A_t^+(t', \Delta^2)$ are evaluated on the hyperbola defined by $s' \cdot u' = \Delta^2$. The hyperbolic dispersion relation for s and u lying on a hyperbola $s \cdot u = b$ may be written as

$$\begin{aligned} T^+(t,b) = \frac{t}{\pi}\int_{4M^2}^{\infty} \frac{d\,t'}{t'}\frac{A_t^+(t',b)}{t'-t} + \frac{1}{\pi}\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'} \\ \times \left[\frac{s}{s'-s} + \frac{u}{s'-u}\right] A_s^+(s', t'_b) + h(b), \end{aligned} \quad (8)$$

where the explicit expression for $h(b)$ may be found in [16]. We do not exhibit it here since this expression only enters the computation for the $t-$ channel partial wave equation, and does not directly enter our considerations. It must also be noted that combining fixed-t and hyperbolic dispersion relations yields an effective dispersion relation on which there are no crossing constraints at a fixed value of $s \cdot u = b$.

For the amplitude $T^-(s,t,u)$ we introduce a new dispersion relation. This is achieved by first considering

$$T^-(s,t,u) = \frac{1}{\pi}\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\left[\frac{s^2}{s'-s}-\frac{u^2}{s'-u}\right] \times A_s^-(s',t) + d(t)(s-u). \quad (9)$$

The subtraction function $d(t)$ is determined by writing down a hyperbolic dispersion relation on $s \cdot u = b$ for $\tilde{T}^-(t,b) = T^-(t,b)/(s-u)$

$$\tilde{T}^-(t,b) = \frac{t}{\pi}\int_{4M^2}^{\infty}\frac{d\,t'}{t'}\frac{\tilde{A}_t^-(t',b)}{t'-t} + \frac{1}{\pi}\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'} \times \left[\frac{s}{s'-s}+\frac{u}{s'-u}\right]\tilde{A}_s^-(s',t'_b) + g(b). \quad (10)$$

We note that these dispersion relations are guaranteed to converge since the fixed-t dispersion is already twice-subtracted and a singly subtracted dispersion relation for $\tilde{T}^-$ is equivalent to a twice-subtracted dispersion relation for T^-. By equating (9) and (10) at $t=0$ and $b=\Delta^2$ we find:

$$d(t) = 2\pi\frac{m+M}{m\,M}\,a_0^- + S^- + L^-(t) + U^-(t),$$

$$S^- = \frac{1}{2\pi}\int_{(m+M)^2}^{\infty}d\,s'\frac{\Delta^2 - s'\Sigma}{s'q_{s'}^2(s'^2-\Delta^2)}A_s^-(s',t'_{\Delta^2}),$$

$$L^-(t) = \frac{t}{\pi}\int_{4M^2}^{\infty}\frac{d\,t'}{t'(t'-t)}\tilde{A}_t^-(t',\Delta^2),$$

$$U^-(t) = \frac{1}{\pi}\int_{(m+M)^2}^{\infty}d\,s'\left[\frac{1}{\Delta^2-s'^2}+\frac{1}{s'(4q_{s'}^2+t)}\right] \times A_s^-(s',t'_{\Delta^2}) + \frac{1}{\pi}\int_{(m+M)^2}^{\infty}d\,s'\left[\frac{1}{s'^2}-\frac{1}{s'(4q_{s'}^2+t)}\right]A_s^-(s',t).$$

The corresponding expression for $g(b)$ may be computed by following the procedure that led to the expressions above, and is not exhibited here since this expression only enters the computation for the $t-$ channel partial wave equation.

4.1 Dispersion relations with $S-$ and $P-$wave absorptive parts

To perform a comparison of the amplitudes $T^\pm$ in their chiral and dispersive framework, we saturate the above fixed-t dispersion relations with S- and P-waves. As it is a straightforward calculation, we give here only two examples of contributions from the dispersion relation for T^+, showing the interplay between chiral and dispersive representations. The integral in (7) can be written as

$$\begin{aligned}
&\frac{1}{\pi}\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\left[\frac{s^2}{s'-s}+\frac{u^2}{s'-u}\right]A_s^+(s',t)\\
&= -12\Delta^2\,(s+u)\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\frac{1}{q_{s'}^2}\mathrm{Im}\,f_1^+(s') + 16\left(s^2+u^2\right)\\
&\times\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\left[\mathrm{Im}\,f_0^+(s') + \frac{3s'}{4q_{s'}^2}\mathrm{Im}\,f_1^+(s')\right]\\
&+16\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\left[\frac{s^3}{s'-s}+\frac{u^3}{s'-u}\right]\mathrm{Im}\,f_0^+(s')\\
&+12\,s^2\left(t-u+\frac{\Delta^2}{s}\right)\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\frac{1}{q_{s'}^2}\frac{\mathrm{Im}\,f_1^+(s')}{s'-s}\\
&+12\,u^2\left(t-s+\frac{\Delta^2}{u}\right)\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\frac{1}{q_{s'}^2}\frac{\mathrm{Im}\,f_1^+(s')}{s'-u}. \quad (11)
\end{aligned}$$

Here one finds polynomials in s and u and integrals which are identical in structure to three of the integrals in (4). The last of the integrals in (4) has a structure whose dispersive counterpart arises from $L^+(t)$. Furthermore, $U^+(t)$ is quadratic in t in the S- and P-wave approximation,

$$\begin{aligned}
&U^+(t) =\\
&-24\,t^2\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2q_{s'}^2}\mathrm{Im}\,f_1^+(s') + 16\,t\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\\
&\times\left(\frac{3(s'^2+6\Sigma s'-\Delta^2)}{4s'q_{s'}^2}\mathrm{Im}\,f_1^+(s') - \mathrm{Im}\,f_0^+(s')\right)\\
&+32\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\left(\Sigma\,\mathrm{Im}\,f_0^+(s')\right.\\
&\left.+\frac{3\left(\Delta^2\Sigma - s'\left(2\left(\Sigma^2-\Delta^2\right)+\Sigma s'\right)\right)}{4s'q_{s'}^2}\mathrm{Im}\,f_1^+(s')\right).
\end{aligned}$$

We take this opportunity to note that the contribution of a state of angular momentum l to U^+ is a polynomial of degree $l+1$, while the contribution to U^- is a polynomial of degree l. However, there does not appear to be any elegant closed form expression for such contributions, which will be of interest in the subsection on contributions from higher waves.

Treating the remaining parts of (7) and (9) in a similar way, the polynomial part of $T^\pm$ can be written, after eliminating the ambiguity associated with the mass shell condition $s+t+u=2\Sigma$ as[3]

$$\begin{aligned}
&T^+_{P,disp.}(s,t,u) =\\
&x_1 + 2\Sigma x_2 + 2\Sigma^2 x_3 + (x_4 - 2\Sigma x_3 - x_2)t\\
&+\frac{1}{2}t^2(x_3+2x_5) + \frac{1}{2}(s-u)^2 x_3, \quad (12)
\end{aligned}$$

[3] The integrals of the dispersive representation of $T^\pm$ are identical to the ones in (4) and (5), respectively.

$$
\begin{aligned}
&T^{-}_{P,disp.}(s,t,u) = \\
&(s-u)\left\{2\pi\frac{m+M}{mM}a_0^- + y_1 + 2\Sigma y_2 + y_3 + y_5\right\} \\
&\quad +t(s-u)\{y_4 - y_2 + y_6\},
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= 8\pi(m+M)a_0^+ + 8\int_{(m+M)^2}^{\infty} d\,s'\frac{\left(\Delta^2-\Sigma\,s'\right)}{s'^2 q_{s'}^2} \\
&\quad \times\left(\mathrm{Im}\,f_0^+(s') - 3\,\mathrm{Im}\,f_1^+(s')\right) \\
&\quad +32\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2}\Bigg(\Sigma\,\mathrm{Im}\,f_0^+(s') \\
&\quad +\frac{3\left[\Delta^2\,\Sigma - s'\left\{2(\Sigma^2-\Delta^2)+\Sigma\,s'\right\}\right]}{4s' q_{s'}^2}\mathrm{Im}\,f_1^+(s')\Bigg), \\
x_2 &= -12\Delta^2\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3 q_{s'}^2}\mathrm{Im}\,f_1^+(s'), \\
x_3 &= 16\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\left[\mathrm{Im}\,f_0^+(s') + \frac{3s'}{4q_{s'}^2}\mathrm{Im}\,f_1^+(s')\right], \\
x_4 &= \frac{16}{\sqrt{3}}\int_{4M^2}^{\infty}\frac{d\,t'}{t'^2}\mathrm{Im}\,f_0^{I_t=0}(t') + 16\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2} \\
&\quad \times\left(\frac{3\left(s'^2+6\,\Sigma\,s'-\Delta^2\right)\,\mathrm{Im}\,f_1^+(s')}{4s' q_{s'}^2} - \mathrm{Im}\,f_0^+(s')\right), \\
x_5 &= \frac{16}{\sqrt{3}}\int_{4M^2}^{\infty}\frac{d\,t'}{t'^3}\mathrm{Im}\,f_0^{I_t=0}(t') \\
&\quad -24\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^2 q_{s'}^2}\mathrm{Im}\,f_1^+(s').
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
y_1 &= \int_{(m+M)^2}^{\infty} d\,s'\frac{-12\Delta^2}{s'^3 q_{s'}^2}\mathrm{Im}\,f_1^-(s'), \\
y_2 &= \int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'}\left[\frac{16}{s'^2}\mathrm{Im}\,f_0^-(s') + \frac{12}{s' q_{s'}^2}\mathrm{Im}\,f_1^-(s')\right], \\
y_3 &= 8\int_{(m+M)^2}^{\infty} d\,s'\frac{\Delta^2-\Sigma s'}{s' q_{s'}^2(s'^2-\Delta^2)} \\
&\quad \times\left(\mathrm{Im}\,f_0^-(s') - 3\mathrm{Im}\,f_1^-(s')\right), \\
y_4 &= 6\sqrt{2}\int_{4M^2}^{\infty}\frac{d\,t'}{t'^2}\mathrm{Im}\,\frac{f_1^{I_t=1}(t')}{p_{t'}q_{t'}}, \\
y_5 &= \int_{(m+M)^2}^{\infty} d\,s'\frac{16\,\Delta^2}{s'^2\left(\Delta^2-s'^2\right)}\mathrm{Im}\,f_0^-(s') + \int_{(m+M)^2}^{\infty} d\,s' \\
&\quad \times\frac{12\left(\Delta^4 - 2\,\Delta^2\,\Sigma\,s' - 3\,\Delta^2\,s'^2 + 4\,\Sigma\,s'^3\right)}{s'^3 q_{s'}^2\left(\Delta^2-s'^2\right)} \\
&\quad \times\mathrm{Im}\,f_1^-(s'), \\
y_6 &= \int_{(m+M)^2}^{\infty} d\,s'\frac{24}{s'^2 q_{s'}^2}\mathrm{Im}\,f_1^-(s').
\end{aligned}
\tag{14}
$$

In the manner described above, we have established the starting point for the comparison of the contributions of S– and P–wave absorptive parts to the low energy polynomial.

4.2 Contributions from higher partial waves

Contributions from higher partial waves are twofold. On the one hand they contribute to polynomials as in (12). On the other hand these waves also yield additional dispersive integrals similar to the last three terms in (11). However, applying the chiral power counting scheme, one can see that the corresponding chiral integrals are of order $O(q^6)$, so that they are neglected in chiral perturbation theory to one loop. Therefore, only the contributions of the higher partial waves to the polynomials in (12) are of interest here. It may be readily seen that the $l \geq 2$ partial waves contribute to the low energy polynomial of T^+at this level as follows: the contribution coming from the fixed-t dispersive integral of (7) to the coefficient of (s^2+u^2) reads

$$
\zeta_{ft} = 16\sum_{l=2}^{\infty}(2l+1)\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\mathrm{Im}\,f_l^+(s').
$$

The contributions to the coefficient of t and t^2 of the polynomial coming from $L^+(t)$ read

$$
\begin{aligned}
\zeta_{L_1} &= \frac{16}{\sqrt{3}}\sum_{l=2}^{\infty}(2l+1)\int_{4M^2}^{\infty}\frac{d\,t'}{t'^2}\mathrm{Im}\,f_l^{I_t=0}(t'), \\
\zeta_{L_2} &= \frac{16}{\sqrt{3}}\sum_{l=2}^{\infty}(2l+1)\int_{4M^2}^{\infty}\frac{d\,t'}{t'^3}\mathrm{Im}\,f_l^{I_t=0}(t'),
\end{aligned}
$$

whereas S^+ contributes to the constant part of the low energy polynomial,

$$
\begin{aligned}
\zeta_S &= 8\sum_{l=2}^{\infty}(-1)^l(2l+1) \\
&\quad \times\int_{(m+M)^2}^{\infty} d\,s'\frac{\left(\Delta^2-\Sigma\,s'\right)}{s'^2 q_{s'}^2}\mathrm{Im}\,f_l^+(s').
\end{aligned}
$$

As noted earlier, the contribution of a partial wave of angular momentum l to U^+ is a polynomial in t of degree $l+1$, and the three lowest coefficients contributing to (12) can be read of from it. Then the expression below is the sum of all such contributions:

$$
\zeta_{U_0} + \zeta_{U_1}t + \zeta_{U_2}t^2.
$$

An analogous procedure for the contributions to T^- from the higher waves may also be performed. The contributions from the fixed-t dispersive integral of (9) will make a contribution proportional to (s^2-u^2) whose coefficient is

$$
\xi_{ft} = 16\sum_{l=2}^{\infty}(2l+1)\int_{(m+M)^2}^{\infty}\frac{d\,s'}{s'^3}\mathrm{Im}\,f_l^-(s').
$$

There is a contribution coming from $L^-(t)$ proportional to t which reads

$$\xi_L = 2\sqrt{2}\sum_{l=3}^{\infty}(2l+1) \times \int_{4M^2}^{\infty} d\,t' \frac{1}{t'^2}\mathrm{Im}\,\frac{f_l^{I_t=1}(t')}{q_{t'}p_{t'}},$$

and a contribution from S^- which reads

$$\xi_S = 8\sum_{l=3}^{\infty}(-1)^l(2l+1) \times \int_{(m+M)^2}^{\infty} d\,s' \frac{\left(\Delta^2 - \Sigma\, s'\right)\,\mathrm{Im}\,f_l^-(s')}{s'^2 q_{s'}^2 (s'-\Delta^2)}.$$

Here we merely denote the sum of such contributions from U^- to the low energy polynomial by

$$\xi_{U^0} + \xi_{U^1} t$$

The resulting polynomials then read

$$\begin{aligned}
&T_{hw}^+(s,t,u) = \\
&2\Sigma^2\zeta_{ft} + \zeta_S + \zeta_{U_0} + (\zeta_{L_1} + \zeta_{U_1} - 2\Sigma\zeta_{ft})t \\
&\quad + \left(\zeta_{L_2} + \zeta_{U_2} + \frac{\zeta_{ft}}{2}\right)t^2 + \frac{\zeta_{ft}}{2}(s-u)^2, \\
&T_{hw}^-(s,t,u) = \\
&(s-u)(2\,\Sigma\,\xi_{ft} + \xi_S + \xi_{U^0} + (\xi_L + \xi_{U^1} - \xi_{ft})t).
\end{aligned} \tag{15}$$

In summary, the dispersive representation for the low energy polynomial is the sum of the contributions arising from the S– and P–wave absorptive parts, (12), and those of the higher partial waves, (15). Once the dispersive representation is saturated with phenomenological absorptive parts and is compared with the chiral representation (6), then the procedure would amount to a determination of the low energy constants of chiral perturbation theory.

4.3 Partial wave equations

Analyticity, unitarity, and crossing symmetry lead to a set of integral equations (Steiner-Roy equations) relating each of the partial waves to all the other ones [4,17,18]. These equations depend on the choice of dispersion relations. In the case at hand, the integral equations are derived by projecting (7)-(10) onto partial waves and by inserting a partial wave expansion for the absorptive parts. For Steiner–Roy equations based on other dispersion relations, see [22–24]. In contrast to the dispersion relations for the full amplitudes, the range of validity of these equations is restricted by the Lehmann ellipse(s). Assuming Mandelstam analyticity, the sum of partial waves for $A^\pm(s,t)$ is convergent for all s if $-32M^2 \le t \le 4M^2$, implying that the partial wave equations for the s-channel waves are valid in the range $2.43M^2 \le s \le 57.14M^2$. Analogously, for the t-channel partial wave equations, the range of validity is $-28.2M^2 \le t \le 82.2M^2$ [4].

The Steiner–Roy equations for the s-channel S- and P-waves from T^+ read (in the S- and P-wave approximation)

$$\begin{aligned}
f_l^+(s) &= \delta_{0,l}\,a_0^+\,\frac{m+M}{2} + \int_{(m+M)^2}^{\infty} d\,s' K_{l,0}^+(s,s') \\
&\quad \times \mathrm{Im}\,f_0^+(s') + \int_{(m+M)^2}^{\infty} d\,s' K_{l,1}^+(s,s')\mathrm{Im}\,f_1^+(s') \\
&\quad + \int_{4M^2}^{\infty} d\,t' K_{l,0}^{(0)}(s,t')\mathrm{Im}\,f_0^{I_t=0}(t'), \\
f_0^{I_t=0}(t) &= \frac{\sqrt{3}}{2}(m+M)a_0^+ + \int_{(m+M)^2}^{\infty} d\,s' G_{0,0}^+(t,s') \\
&\quad \times \mathrm{Im}\,f_0^+(s') + \int_{(m+M)^2}^{\infty} d\,s' G_{0,1}^+(t,s')\mathrm{Im}\,f_1^+(s') \\
&\quad + \int_{4M^2}^{\infty} d\,t' G_{0,0}^{(0)}(t,t')\mathrm{Im}\,f_0^{I_t=0}(t'),
\end{aligned} \tag{16}$$

while the ones obtained from T^- are

$$\begin{aligned}
&f_l^-(s) = \\
&\delta_{l,0}\,a_0^-\,\frac{(m+M)}{2}\,\frac{3s^2 - 2s(m^2+M^2) - (m^2-M^2)^2}{8\,s\,m\,M} \\
&\quad + \delta_{l,1}a_0^-\,\frac{(m+M)}{2}\,\frac{m^4 + (M^2-s)^2 - 2m^2(M^2+s)}{24\,s\,m\,M} \\
&\quad + \int_{(m+M)^2}^{\infty} d\,s' K_{l,0}^-(s,s')\mathrm{Im}\,f_0^-(s') \\
&\quad + \int_{(m+M)^2}^{\infty} d\,s' K_{l,1}^-(s,s')\mathrm{Im}\,f_1^-(s') \\
&\quad + \int_{4M^2}^{\infty} d\,t' K_{l,1}^{(1)}(s,t')\mathrm{Im}\,\frac{f_1^{I_t=1}(t')}{q_{t'}p_{t'}}, \\
&f_1^{I_t=1}(t) = \\
&a_0^-\,\frac{(m+M)}{2}\,\frac{\sqrt{t-4m^2}\sqrt{t-4M^2}}{6\sqrt{2}m\,M} \\
&\quad + \int_{(m+M)^2}^{\infty} d\,s' G_{1,0}^-(t,s')\mathrm{Im}\,f_0^-(s') \\
&\quad + \int_{(m+M)^2}^{\infty} d\,s' G_{1,1}^-(t,s')\mathrm{Im}\,f_1^-(s') \\
&\quad + \int_{4M^2}^{\infty} d\,t' G_{1,1}^{(1)}(t,t')\mathrm{Im}\,\frac{f_1^{I_t=1}(t')}{q_{t'}p_{t'}}.
\end{aligned} \tag{17}$$

The kernel functions K, G can be found in Appendix C.

We note here that this is effectively a system of closed equations for the S– and P–waves. In order to solve them in the low-energy region, in practice the contributions of the absorptive parts of the $l \ge 2$ waves and that of the high energy tail of the S– and P–waves are added together to yield the driving terms for this system from the dispersion relations (7)-(10), when expressions for the absorptive parts of the $l \ge 2$ are inserted into the right hand sides and by writing down forms for the S– and P–waves compatible with unitarity and with the requirement that they reproduce the scattering lengths. Such a program has been recently carried out for $\pi\pi$ scattering, see [10].

5 Low energy constants from phenomenology

The coefficients of the chiral polynomials, (6), are functions of the low energy constants L_i^r whereas the coefficients of the dispersive polynomial part of the amplitudes, (12), are given in terms of integrals over the lowest six partial waves. Once the imaginary parts of these are known, a comparison with the chiral polynomials yields the low energy constants involved in πK scattering. As (6) and (12) only provide six equations, only constraints for combinations of the seven low energy constants involved can be derived.

In the present work we focus on the combinations $4L_2^r + L_3$, $4L_1^r + L_3 - 4L_4^r - L_5^r + 4L_6^r + 2L_8^r$, and $F_\pi F_K + 4L_5^r(M^2 + m^2)$. The first appears in the term proportional to $(s-u)^2$ in T^+, while the second one comes from the constant part of the same amplitude. The last combination stems from the t-independent part of the amplitude T^-. Only $4L_2^r + L_3$ does not explicitly depend on the scattering lengths. Therefore we expect this combination to be the one which can be estimated most precisely.

To evaluate the coefficients of the dispersive polynomials, (13,14), we employ a K–matrix parametrization similar to the ones in [25], but with few more free parameters, and require the resulting phase shifts to fit the experimental data of [26] in the elastic region. As the integral cut–offs in (13) and (14) we choose the elasticity thresholds $1.69\mathrm{GeV}^2$ for $I = 1/2$ and $2.96\ \mathrm{GeV}^2$ for $I = 3/2$. Here, we do not take into account the contributions from the higher partial waves. The masses of the pion, kaon, and eta, the latter entering the calculation only through the loop–functions, are set to $M = 139.56$ MeV, $m = 497.67$ MeV, and $m_\eta = 547.30$ MeV, respectively. The decay constants are $F_\pi = 92.4$ MeV, $F_K = 1.22F_\pi$ (we take the well-established analysis for the ratio F_K/F_π in the present work; new analyses are now available [27], and these will be incorporated at the time the fresh Steiner-Roy equation fits to the data are ready [28]). Furthermore, the renormalization scale μ is set to $m_\rho = 769.30$ MeV. For the three combinations of low energy constants we obtain

$$\begin{aligned}
&4L_2^r + L_3^r = 0.0027 \pm 0.0001, \\
&4L_1^r + L_3^r - 4L_4^r - L_5^r + 4L_6^r + 2L_8^r \\
&\quad = -0.0003 \pm 0.0013 + 0.14\,\mathrm{GeV} \cdot a_0^+, \\
&L_5^r = -0.0065 \pm 0.0001 + 0.024\,\mathrm{GeV} \cdot a_0^-,
\end{aligned} \tag{18}$$

where $a_0^\pm$ are given in GeV^{-1}. The quoted errors are due to varying the integral cut–offs by 20%. Note that the coefficients of the scattering lengths, i.e. $0.14\,\mathrm{GeV}^{-1}$ and $0.024\,\mathrm{GeV}^{-1}$, are fixed by chiral perturbation theory.

In order to check the influence of the parametrization on the above numerical results, we have chosen yet another K–matrix parametrization with fewer parameters to fit the experimental data. The quality of these fits is not as good, especially for the $I = 1/2$ S–wave and the $I = 3/2$ P–wave, whereas the other two waves are not changed significantly. However, as the phases of the $I = 3/2$ P–wave are small at low energies, changes to this partial wave are not important. This parametrization yields

$$\begin{aligned}
&4L_2^r + L_3^r = 0.0029 \pm 0.0001, \\
&4L_1^r + L_3^r - 4L_4^r - L_5^r + 4L_6^r + 2L_8^r \\
&\quad = -0.012 \pm 0.001 + 0.14\,\mathrm{GeV} \cdot a_0^+, \\
&L_5^r = -0.012 \pm 0.00001 + 0.024\,\mathrm{GeV} \cdot a_0^-,
\end{aligned} \tag{19}$$

where again the quoted errors are due to changes in the integration cut–offs. The numerical coefficients on the right hand sides of (18,19) generally depend on the renormalization scale μ, as well. Note that the last lines in (18,19) amount to an alternative method to fix L_5^r, which could then be employed to determine the ratio of the pion and the kaon decay constants. However, as the predictions of the decay constants are much more reliable than the one for the πK scattering length, we regard the relation between L_5^r and a_0^- as a consistency check for our method than a accurate way of determining L_5^r. Comparing the above with the values in [29],

$$\begin{aligned}
&4L_2^r + L_3 = 0.0019 \pm 0.0013, \\
&4L_1^r + L_3 - 4L_4^r - L_5^r + 4L_6^r + 2L_8^r \\
&\quad = -0.0011 \pm 0.0018 \\
&L_5^r = 0.0014 \pm 0.0005,
\end{aligned} \tag{20}$$

one can see that our values for the first combination are in reasonable agreement with the previous determination. The values for the second combination of low energy constants are still reasonable if the wide spread of experimental values for the scattering lengths (see [2] and references therein),

$$\begin{aligned}
-0.31\,\mathrm{GeV}^{-1} &\leq a_0^+ \leq 0.34\,\mathrm{GeV}^{-1}, \\
0.43\,\mathrm{GeV}^{-1} &\leq a_0^- \leq 0.89\,\mathrm{GeV}^{-1},
\end{aligned}$$

is taken into account, whereas the determination of L_5^r is less reliable, emphasizing the need of a detailed analysis of the πK partial waves and the importance of the contributions of the higher partial waves to the low energy polynomials. This will be done elsewhere [28]. Furthermore, there are other recent determinations of the low energy constants [30] and a detailed comparison will be made when the Steiner-Roy equation fits are available [28].

Comparing the results in (18,19), one can see that the numerical value for the first of the above combinations for the LECs does not depend very much on the parametrization, whereas this dependence for the second and the third combination is more substantial. Note, however, that this dependence is accommodated by the wide spread of experimental values for the scattering lengths, again calling for a more detailed analysis of the phase shifts and the scattering lengths.

6 Summary and conclusions

The πK scattering problem is an important process in the low energy sector of the strong interactions. Compared to

Table 1. $SU(3)$ coupling constants from $\pi\pi$ phase shifts ($\mu = m_\rho$)

	Set 1	Set 2	Set 3
$10^3 L_2^r$	1.63	1.63	1.62
$10^3(2L_1^r + L_3)$	−2.34	−2.28	−2.22
$10^3(2L_4^r + L_5^r)$	−1.92	−1.56	−1.23

the closely related problem of $\pi\pi$ scattering, considerably less is known about this process due to the lack of availability of experimental data, the relative paucity of theoretical results associated with the absence of three-channel crossing symmetry, and the presence of unequal masses of the particles. Despite these difficulties, here we have shown that the results of the type established in the recent past for $\pi\pi$ scattering can be extended to the πK case. We have noted that the experimentally accessible scattering length a_0^- at one-loop order in chiral perturbation theory is essentially parameter free and that a precise determination of this quantity would constitute a precision test of chiral perturbation theory.

We have established a framework within which the πK amplitude in $SU(3)$ chiral perturbation theory can be split up into functions of single variables and are then replaced by an dispersive representation that leads to an effective low energy polynomial representation and a dispersive tail. We have considered twice–subtracted fixed-t dispersion relations with the subtraction functions determined in terms of dispersion relations on hyperbolic curves, in particular those for T^+ which were established sometime ago, and new ones for T^-. This allows us to generate a low energy polynomial and a dispersive tail with the same structure as the chiral representation. We have also discussed in some detail the contributions of the absorptive parts of the $l \geq 2$ partial waves.

Furthermore, explicit integral equations for the S– and P–waves are given which form a closed system when the $l \geq 2$ wave absorptive parts are neglected. The contributions from those waves and the high energy part of the S– and P–wave absorptive parts would then determine the driving terms for these (Steiner-Roy) integral equations. In particular, a detailed fit of experimental information to these equations would lead to a precise determination of a_0^-.

The comparison of chiral perturbation and dispersion theory representation of the amplitudes yields a system of sum rules for low energy constants of chiral perturbation theory. We have used the results from a recent study of the phase shift and elasticity information to evaluate certain combinations of coupling constants, which do not involve the contributions coming from the t-channel absorptive parts. First estimates for the $SU(3)$ low energy constants obtained from this phenomenology yield the estimates, see (18). These estimates compare favorably with the determinations reported in the literature. A full partial wave analysis will lead to an accurate evaluation of the coupling constants of interest. Such a partial wave analysis combined with chiral inputs can produce reliable estimates for πK scattering lengths which can in principle be measured at pion-kaon atom "factories" such as DIRAC.

Acknowledgements. We are particularly indebted to H. Leutwyler and U.-G. Meißner for discussions and comments on the manuscript. We also thank B. Moussallam, J. A. Oller, J. P. Peláez, and J. Stahov for discussions and comments. B. A. thanks the Institut für Kernphysik, Forschungszentrum Jülich, for its hospitality when part of this work was done. P. B. thanks the Centre for Theoretical Studies, Indian Institute of Science, Bangalore, for its hospitality when this work was initiated.

A $\pi\pi$ amplitude in $SU(3)$ chiral perturbation theory

In this brief appendix, we consider the $\pi\pi$ scattering amplitude presented in [2,3]. It is possible to split the amplitude $A(s,t,u)$ into three functions of one variable $W_i(s)$, $i = 1,2,3$, whose absorptive parts may be expressed in terms of those of the three lowest partial waves $f_0^I, I = 0,2$ and f_1^1. In terms of these functions, we may write $A(s,t,u)$ as

$$A(s,t,u) = 32\pi \left\{ \frac{1}{3}W_0(s) + \frac{3}{2}(s-u)W_1(t) + \frac{3}{2}(s-t) \right.$$
$$\left. \times W_1(u) + \frac{1}{2}\left(W_2(t) + W_2(u) - \frac{2}{3}W_2(s) \right) \right\}.$$

We list one choice of for the functions W_i, $i = 0,1,2$ in Appendix B. One may now write the W_i in terms of dispersion relations and generate a low energy polynomial representation. As an illustration we use the dispersive polynomial established in [9] to evaluate the $SU(3)$ low energy constants with the three sets of phase shifts described there. These results are presented in Table 1 (masses, decay constants and renormalization scale as in Sect. 5).

We note that these phase shifts were used to determine the values for the low energy constants $\bar{l}_i$, $i = 1,2,4$ of $SU(2)$ chiral perturbation theory. It is well known that when the SU(3) theory is reduced to SU(2) theory, relations emerge between the low energy constants of the two theories. The results for the $\bar{l}_i$, $i = 1,2,4$ of [9] may then be translated into the SU(3) coupling constants which are listed in Table 2. Although the results of Table 1 and Table 2 are in general agreement, those in Table 1 amount to a consistent new determination. The numbers in Table 1 agree well with determinations in the literature, see, e.g., [1,29].

The phase shift determination of [9] were based on a Roy equation fit whose driving terms were computed from higher wave and asymptotic contributions that arose from the $f_2(1270)$ and Pomeron and Regge contributions setting in above an energy of ~ 1.5 GeV, recently described in [31]. These absorptive parts also contribute to the low energy dispersive polynomials. We evaluate the resulting contributions to the low energy constants whose contributions to $\bar{l}_1$ and $\bar{l}_2$ are ~ -0.1 and 0.41, respectively and

Table 2. $SU(3)$ coupling constants derived from $SU(2)$ effective theory ($\mu = m_\rho$)

	Set 1	Set 2	Set 3
$10^3 L_2^r$	1.42	1.41	1.41
$10^3(2L_1^r + L_3)$	−2.16	−2.10	−2.04
$10^3(2L_4^r + L_5^r)$	−1.55	−1.20	−0.88

to $10^3\, L_2^r$ and $10^3(2L_1^r + L_3)$ are ~ 0.21 and -0.05 respectively. We note here that once the scattering length a_0^0 is experimentally determined to within small uncertainties, the Roy equation fits of [10] may be used to produce sharp values for the combinations of low energy constants discussed here.

B List of functions of single variables

The functions of single variables entering $T^+(s,t,u)$ are given as

$$
\begin{aligned}
&16F_\pi^2 F_K^2 Z_0^+(s) = \\
&\frac{3\,\Delta^2(L_{K\eta}(s) + L_{\pi K}(s))}{s} + 2\,\Delta\Sigma(K_{K\eta}(s) + K_{\pi K}(s)) \\
&+\Sigma^2\left(\frac{J_{K\eta}^r(s)}{3} + 3\,J_{\pi K}^r(s)\right) - s\,\Big(4\,F_K\,F_\pi + \Delta(3\,K_{K\eta}(s) \\
&+5\,K_{\pi K}(s)) + 16\Sigma(8\,L_2^r + 2\,L_3 + L_5^r) + \Sigma(J_{K\eta}^r(s) \\
&+7\,J_{\pi K}^r(s))\Big) + s^2\,\left(64\,L_2^r + 16\,L_3 + \frac{3\,J_{K\eta}^r(s)}{4}\right. \\
&\left.+\frac{19\,J_{\pi K}^r(s)}{4}\right),
\end{aligned}
$$

$$
\begin{aligned}
&16F_\pi^2 F_K^2 Z_1^+(s) = \\
&-3\,\left(L_{K\eta}(s) + L_{\pi K}(s) - s\,\left(M_{K\eta}^r(s) + M_{\pi K}^r(s)\right)\right), \\
&16F_\pi^2 F_K^2 Z_t^+(s) = \\
&8\,F_K\,F_\pi\,\Sigma + 128 M^2 m^2\Big(4\,L_1^r + L_3 - 4\,L_4^r - L_5^r \\
&+4\,L_6 + 2\,L_8\Big) - \frac{16 M^2 m^2 J_{\eta\eta}^r(s)}{9} + F_K\,F_\pi \Delta(3\,\mu_\pi \\
&-2\,\mu_K - \mu_\eta) + 32\Sigma^2(4\,L_2^r + L_3 + L_5^r) \\
&-s\,\Big(64\Sigma(4\,L_1^r + L_3 - 2\,L_4^r) + 2\,M^2(J_{\pi\pi}^r(s) - J_{\eta\eta}^r(s))\Big) \\
&+s^2\,\left(32(4\,L_1^r + L_3) + 3\,J_{KK}^r(s) + 4\,J_{\pi\pi}^r(s)\right),
\end{aligned}
$$

whereas the functions of single variables entering $T^-(s,\,t,\,u)$ read

$$
\begin{aligned}
&96 F_K^2 F_\pi^2 Z_0^-(s) = \\
&\frac{18\,\Delta^2(L_{K\eta}(s) + L_{\pi K}(s))}{s} + 12\,\Delta\Sigma(K_{K\eta}(s) + K_{\pi K}(s)) \\
&+2\,\Sigma^2(J_{K\eta}^r(s) - 3\,J_{\pi K}^r(s)) + s\,\Big(24\,F_K\,F_\pi - 6\Delta(3\,K_{K\eta}(s) \\
&+5\,K_{\pi K}(s)) - 96\Sigma(2\,L_3 - \,L_5^r) - 6\Sigma(J_{K\eta}^r(s) - J_{\pi K}^r(s))\Big) \\
&+s^2\,\left(96\,L_3 + \frac{9}{2}(J_{K\eta}^r(s) + J_{\pi K}^r(s))\right),
\end{aligned}
$$

$$
\begin{aligned}
&96 F_K^2 F_\pi^2 Z_1^-(s) = \\
&-18\,(s\,M_{K\eta}^r(s) + s\,M_{\pi K}^r(s) - L_{K\eta}(s) - L_{\pi K}(s)),
\end{aligned}
$$

$$
96 F_K^2 F_\pi^2 Z_t^-(s) = 24\,s\,(M_{KK}^r(s) + 2 M_{\pi\pi}^r(s)).
$$

Finally, the functions of single variables required to define the $\pi\pi$ amplitude in $SU(3)$ chiral perturbation theory can be written as

$$
\begin{aligned}
W_0(s) = \frac{3}{32\pi}\Bigg\{ &\frac{s - M^2}{F_\pi^2} + \frac{1}{F_\pi^4}\left(\frac{M^4}{18} J_{\eta\eta}^r(s) + \frac{1}{2}(s^2 - M^4)^2\right. \\
&\left.\times J_{\pi\pi}^r(s) + \frac{s^2}{8} J_{KK}^r(s)\right) + \frac{4}{F_\pi^4}\Big[(2L_1^r + L_3) \\
&\times(s - 2M^2)^2 + (4L_4^r + 2L_5^r)\,(s - 2M^2)\,M^2 \\
&+(8L_6^r + 4L_8^r)M^4\Big]\Bigg\} + W_2(s)
\end{aligned}
$$

$$
W_1(s) = \frac{s}{48\pi}\left\{M_{\pi\pi}^r(s) + \frac{1}{2} M_{KK}^r(s)\right\}
$$

$$
W_2(s) = \frac{1}{16\pi}\left\{\frac{1}{4F_\pi^4}(s - 2M^2)^2 J_{\pi\pi}^r(s) + \frac{4L_2^r}{F_\pi^4}(s - 2M^2)^2\right\}
$$

For the definitions of the standard loop functions $J_{PQ}^r(s)$, $M_{PQ}^r(s)$, $L_{PQ}(s)$, $K_{PQ}(s)$ and μ_P, see [1].

C Kernels of the partial wave equations

The kernels of the partial wave equations, (16,17), are:

$$
\begin{aligned}
&K_{0,0}^+(s,s') = \\
&\frac{s' - 2\,s}{\pi\,(s - s')\,s'} + \frac{\Delta^2 - \Sigma\,s'}{2\,\pi\,s'^2\,{q_{s'}}^2} \\
&+\frac{\ln(s - 2\,\Sigma + s') - \ln(s - 2\,\Sigma + s' - 4\,{q_s}^2)}{4\,\pi\,{q_s}^2},
\end{aligned}
$$

$$
\begin{aligned}
&K_{01}^+(s,s') = \\
&\Big\{\Big(-3\,\left(\ln(s - 2\,\Sigma + s') - \ln(s - 2\,\Sigma + s' - 4\,{q_s}^2)\right) \\
&\times\left(s - 2\,\Sigma + s' - 2\,{q_{s'}}^2\right)\Big)\Big/\left(8\,\pi\,{q_s}^2\,{q_{s'}}^2\right)\Big\} \\
&+\Big\{\Big(3\,{q_s}^2\,\Big\{\Delta^2\,(s - s') - s'\,\left(2\,s\,\Sigma - 3\,s\,s' - 2\,\Sigma\,s' + s'^2\right) \\
&+4\,s'\,(s' - s)\,{q_{s'}}^2\Big\}\Big)\Big/\left(2\,\pi\,(s - s')\,s'^3\,{q_{s'}}^2\right)\Big\} \\
&+\Big\{\Big(3\,\Big\{(s - s')\,\left(s'^3 + \Delta^2\,(\Sigma + s') - 2\,\Sigma^2\,s'\right) \\
&+2\,s'\,[s'\,(2\,\Sigma + s') - 2\,s\,(\Sigma + s')]\,{q_{s'}}^2\Big\}\Big)\Big/ \\
&\left(2\,\pi\,(s - s')\,s'^3\,{q_{s'}}^2\right)\Big\},
\end{aligned}
$$

$$
K_{0,0}^{(0)}(s,t') = \frac{t'\,\left\{\ln(t' + 4\,{q_s}^2) - \ln(t')\right\} - 4\,{q_s}^2}{4\,\sqrt{3}\,\pi\,t'\,{q_s}^2},
$$

$$K^{+}_{1,0}(s,s') = \frac{1}{2\,\pi\,{q_s}^2} - \Big\{ \Big(\{\ln(s-2\,\Sigma+s') - \ln(s-2\,\Sigma+s' -4\,{q_s}^2)\}\ \left(s-2\,\Sigma+s'-2\,{q_s}^2\right) \Big) \Big/ \left(8\,\pi\,{q_s}^4\right) \Big\},$$

$$K^{+}_{1,1}(s,s') = \Big\{ \Big(-3\ \left(s-2\,\Sigma+s'-2\,{q_{s'}}^2\right) \Big) \Big/ \left(4\,\pi\,{q_s}^2\,{q_{s'}}^2\right) \Big\} + \Big\{ \Big(3\ \{\ln(s-2\,\Sigma+s') - \ln(s-2\,\Sigma+s'-4\,{q_s}^2)\} \times \left(s-2\,\Sigma+s'-2\,{q_s}^2\right)\ \left(s-2\,\Sigma+s'-2\,{q_{s'}}^2\right) \Big) \Big/ \left(16\,\pi\,{q_s}^4\,{q_{s'}}^2\right) \Big\} - \Big\{ \Big({q_s}^2\ \Big(\Delta^2\ (s-s') - s'\ \left(2\,s\,\Sigma - 3\,s\,s' - 2\,\Sigma\,s' + {s'}^2\right) + 4\,s'\ (s'-s)\ {q_{s'}}^2\Big)\Big) \Big/ \Big(2\,\pi\ (s-s')\ {s'}^3\,{q_{s'}}^2\Big) \Big\},$$

$$K^{(0)}_{1,0}(s,t') = \Big\{ \Big(t'\ \{\ln(t'+4\,{q_s}^2) - \ln(t')\} - 2\ \{2+\ln(t') - \ln(t'+4\,{q_s}^2)\}\ {q_s}^2 \Big) \Big/ \left(8\,\sqrt{3}\,\pi\,{q_s}^4\right) \Big\},$$

$$G^{+}_{0,0}(t,s') = \frac{\sqrt{3}}{\pi}\ \left(\frac{\Sigma - s'}{2s'q_{s'}^2} + \frac{4}{\sqrt{4\,m^2-t}\,\sqrt{4\,M^2-t}} \times\ \mathrm{arccoth}\left(\frac{t-2\ (\Sigma-s')}{\sqrt{4\,m^2-t}\,\sqrt{4\,M^2-t}}\right)\right),$$

$$G^{+}_{0,1}(t,s') = \frac{3\,\sqrt{3}\ (\Sigma-s'-t)}{2\,\pi\,s'q_{s'}^2} + \frac{6\,\sqrt{3}\ \left(2s'q_{s'}^2 +\ s'\,t\right)}{\pi\,s'q_{s'}^2\ \sqrt{4\,m^2-t}\,\sqrt{4\,M^2-t}} \times \mathrm{arccoth}\left(\frac{t-2\ (\Sigma-s')}{\sqrt{4\,m^2-t}\,\sqrt{4\,M^2-t}}\right),$$

$$G^{(0)}_{0,0}(t,t') = \frac{t}{\pi t'(t'-t)},$$

$$K^{-}_{0,0}(s,s') = \Big\{ \Big(-\left(3\,s^3 - 3\,s^2\,s' + 2\,s\,{s'}^2 + s\,\Delta^2 + s'\,\Delta^2 - 2\,s\ (s+s')\ \Sigma\right) \Big) \Big/ \Big(2\,\pi\,s\left(s-s'\right) \times \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right) \Big) \Big\} + \frac{s\ \{\ln(s\,s'-\Delta^2) - \ln(s\ [s+s'-2\,\Sigma])\}}{\pi\ (s^2+\Delta^2-2\,s\,\Sigma)},$$

$$K^{-}_{0,1}(s,s') = \frac{-3\ \{\Delta^2 + s\ (3\,s+2\,s'-2\,\Sigma)\}}{2\,\pi\,s\ \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right)} - \Big\{ \Big(3\,s\ \left(s'\ (2\,s+s'-2\,\Sigma) - \Delta^2\right)\ \{\ln(s\,s'-\Delta^2) - \ln(s\ [s+s'-2\,\Sigma])\} \Big) \Big/ \Big(\pi\ \left(s^2+\Delta^2-2\,s\,\Sigma\right) \times \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right) \Big) \Big\},$$

$$K^{(1)}_{0,1}(s,t') = \frac{3\ \{\Delta^2 - s\ (3\,s+2\,t'-2\,\Sigma)\}}{8\,\sqrt{2}\,\pi\,s\,t'} - \Big\{ \Big(3\,s\ (2\,s+t'-2\,\Sigma) \times \{\ln(-2\,s\,t') - \ln(-2\ \left[\Delta^2 + s\ (s+t'-2\,\Sigma)\right])\} \Big) \Big/ \Big(4\,\sqrt{2}\,\pi\ \left(s^2+\Delta^2-2\,s\,\Sigma\right) \Big) \Big\},$$

$$K^{-}_{1,0}(s,s') = \Big\{ \Big(-\Big\{ s^4 + \Delta^4 - 4\,s^3\,\Sigma - 4\,s\,\Delta^2\,\Sigma + 2\,s^2 \times \left(6\,{s'}^2 + 7\,\Delta^2 - 12\,s'\,\Sigma + 2\,\Sigma^2\right) \Big\} \Big) \Big/ \Big(6\,\pi\,s\ \left(s^2+\Delta^2 - 2\,s\,\Sigma\right)\ \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right) \Big) \Big\} + \Big\{ \Big(s\ \{\Delta^2 - s\ (s+2\,s'-2\,\Sigma)\}\ \{\ln(s\,s'-\Delta^2) - \ln(s\ [s+s'-2\,\Sigma])\} \Big) \Big/ \Big(\pi\left(s^2+\Delta^2-2\,s\,\Sigma\right)^2 \Big) \Big\},$$

$$K^{-}_{1,1}(s,s') = \frac{(s+s')}{2\,\pi\,s'\,(s'-s)} + \frac{6\ \left(s^2+s\,s'\right)}{\pi\ (s'-s)\ (s^2+\Delta^2-2\,s\,\Sigma)} + \frac{s^3\,s' - 24\,s^2\,{s'}^2 - s\,{s'}^3 - s^2\,\Delta^2 + {s'}^2\,\Delta^2}{2\,\pi\,s\,s'\ (s'-s)\ \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right)} + \Big\{ \Big(3\,s\ \{\Delta^2 - s'\ (2\,s+s'-2\,\Sigma)\} \times \{\Delta^2 - s\ (s+2\,s'-2\,\Sigma)\}\ \{\ln(s\,s'-\Delta^2) - \ln(s\ [s+s'-2\,\Sigma])\} \Big) \Big/ \Big(\pi\left(s^2+\Delta^2-2\,s\,\Sigma\right)^2 \times \left({s'}^2+\Delta^2-2\,s'\,\Sigma\right) \Big) \Big\},$$

$$K^{(1)}_{1,1}(s,t') = \Big\{ \Big(-\Big(s^4 + \Delta^4 + 4\,s^3\ (6\,t'-\Sigma) - 4\,s\,\Delta^2\,\Sigma + 2\,s^2$$

$$\times \left(6\,t'^2 + \Delta^2 - 12\,t'\,\Sigma + 2\,\Sigma^2\right)\Big)\Bigg) \Bigg/$$
$$\left(8\,\sqrt{2}\,\pi\,s\,t'\,\left(s^2 + \Delta^2 - 2\,s\,\Sigma\right)\right)\Bigg\}$$
$$-\Bigg\{\Bigg(3\,s\,\left\{\Delta^2 + s\,(s + 2\,t' - 2\,\Sigma)\right\}\{2\,s + t' - 2\,\Sigma\}$$
$$\times \left\{\ln(-2\,s\,t') - \ln(-2\,\left[\Delta^2 + s\,(s + t' - 2\,\Sigma)\right])\right\}\Bigg)\Bigg/$$
$$\left(4\,\sqrt{2}\,\pi\,\left(s^2 + \Delta^2 - 2\,s\,\Sigma\right)^2\right)\Bigg\},$$

$$G^-_{1,0}(t,s') =$$
$$\Big\{\Big(-\sqrt{2}\,\Big\{8\,\Delta^2 + 4\,\Sigma^2 + 12\,s'^2 + t^2$$
$$-4\,\Sigma\,(6\,s' + t)\Big\}\Big)\Big/\Big(3\,\pi\,\left\{\Delta^2 + s'\,(-2\,\Sigma + s')\right\}$$
$$\times\sqrt{t + 2\,\Delta - 2\,\Sigma}\sqrt{t - 2\,\Delta - 2\,\Sigma}\Big)\Big\}$$
$$+\frac{4\,\sqrt{2}\,(2\,s' + t - 2\,\Sigma)}{\pi\,(2\,(\Delta + \Sigma) - t)\,\sqrt{(t + 2\,\Delta - 2\,\Sigma)^2}}$$
$$\times\mathrm{arccoth}\left(i\,\frac{2\,\Sigma - 2\,s' - t}{\sqrt{2\,\Sigma - 2\,\Delta - t}\,\sqrt{-2\,\Sigma - 2\,\Delta + t}}\right),$$

$$G^-_{1,1}(t,s') =$$
$$-\Big\{\Big(\sqrt{2}\,\Big\{8\,\Delta^2 + 4\,\Sigma^2 + 12\,s'^2 + 24\,s'\,t + t^2$$
$$-4\,\Sigma\,(6\,s' + t)\Big\}\Big)\Big/\Big(\pi\,\left\{\Delta^2 + s'\,(-2\,\Sigma + s')\right\}$$
$$\times\sqrt{t + 2\,\Delta - 2\,\Sigma}\,\sqrt{t - 2\,\Delta - 2\,\Sigma}\Big)\Big\}$$
$$+\Big\{\Big(12\,\sqrt{2}\,(2\,s' + t - 2\,\Sigma)\,\left\{\Delta^2 + s'\,(s' + 2\,t - 2\,\Sigma)\right\}\Big)\Big/$$
$$\Big(\pi\,\left\{\Delta^2 + s'\,(s' - 2\,\Sigma)\right\}\,(2\,(\Delta + \Sigma) - t)$$
$$\times\sqrt{(t + 2\,\Delta - 2\,\Sigma)^2}\Big)\Big\}$$
$$\times\mathrm{arccoth}\left(i\,\frac{2\,\Sigma - 2\,s' - t}{\sqrt{2\,\Sigma - 2\,\Delta - t}\,\sqrt{-2\,\Sigma - 2\Delta + t}}\right),$$

$$G^{(1)}_{1,1}(t,t') = \frac{t\,\sqrt{t - 2\,\Delta - 2\,\Sigma}\,\sqrt{t + 2\,\Delta - 2\,\Sigma}}{4\,\pi\,t'\,(t' - t)}.$$

References

1. J. Gasser, H. Leutwyler, Nucl. Phys. **B250** (1985) 465
2. V. Bernard, N. Kaiser, U.-G. Meißner, Nucl. Phys. **B357** (1991) 129
3. A. Dobado, J. R. Peláez, Phys. Rev. **D56** (1997) 3057 [hep-ph/9604416]
4. C. B. Lang, Fortsch. Phys. **26** (1978) 509
5. J. Gasser, H. Leutwyler, Annals Phys. **158** (1984) 142
6. M. Knecht, B. Moussallam, J. Stern, N. H. Fuchs, Nucl. Phys. **B457** (1995) 513 [hep-ph/9507319]; M. Knecht, B. Moussallam, J. Stern, N. H. Fuchs, Nucl. Phys. **B471** (1996) 445 [hep-ph/9512404]
7. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M. E. Sainio, Nucl. Phys. **B508** (1997) 263 [hep-ph/9707291]; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M. E. Sainio, Phys. Lett. **B374** (1996) 210 [hep-ph/9511397]
8. J. Stern, H. Sazdjian, N. H. Fuchs, Phys. Rev. **D47** (1993) 3814 [hep-ph/9301244]
9. B. Ananthanarayan, P. Büttiker, Phys. Rev. **D54** (1996) 1125 [hep-ph/9601285]
10. B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, hep-ph/0005297
11. G. Colangelo, J. Gasser, H. Leutwyler Phys. Lett. **B488** (2000) 261 [hep-ph/0007112]
12. J. Stern, hep-ph/9801282; B. Moussallam, Eur. Phys. J. **C14** (2000) 111 [hep-ph/9909292]
13. B. Adeva et al., CERN/SPSC 2000-032, SPSC/P284 Add. 1
14. U.-G. Meißner, private communication
15. M. Knecht, H. Sazdjian, J. Stern, N. H. Fuchs, Phys. Lett. **B313** (1993) 229 [hep-ph/9305332]
16. N. Johannesson, G. Nilsson, Nuovo Cim. **A43** (1978) 376
17. F. Steiner, Fortsch. Phys. **19** (1971) 115; Fortsch. Phys. **18** (1970) 43
18. S. M. Roy, Phys. Lett. **B36** (1971) 353
19. A. Karabarbounis, G. Shaw, J. Phys. G **G6** (1980) 583
20. B. Ananthanarayan, P. Büttiker, work in progress
21. A. Roessl, Nucl. Phys. **B555** (1999) 507 [hep-ph/9904230]
22. H. Nielsen, G. C. Oades, Nucl. Phys. **B55** (1973) 301
23. N. Hedegaard-Jensen, Nucl. Phys. **B77** (1974) 173
24. N. O. Johannesson, J. L. Petersen, Nucl. Phys. **B68** (1974) 397
25. M. Jamin, J. A. Oller, A. Pich, Nucl. Phys. **B587** (2000) 331 [hep-ph/0006045]
26. P. Estabrooks, R. K. Carnegie, A. D. Martin, W. M. Dunwoodie, T. A. Lasinski, D. W. Leith, Nucl. Phys. **B133** (1978) 490
27. N. H. Fuchs, M. Knecht, J. Stern, Phys. Rev. **D62** (2000) 033003 [hep-ph/0001188]
28. B. Ananthanarayan, P. Büttiker, U.-G. Meißner, work in progress
29. J. Bijnens, G. Ecker, J. Gasser, hep-ph/9411232, published in [32], pp. 125
30. G. Amoros, J. Bijnens, P. Talavera, Nucl. Phys. **B585** (2000) 293 [hep-ph/0003258]
31. B. Ananthanarayan, P. Büttiker, Phys. Rev. **D54** (1996) 5501 [hep-ph/9604217]
32. L. Maiani, G. Pancheri, N. Pancheri, *The Second* DAΦNE *Physics Handbook* (INFN-LNF-Divsione Ricerca, SIS-Ufficio Publicazioni, Frascati, 1995)